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The volkenborn integral of the p-adic gamma function

Özge Çolakoğlu Havare *, Hamza Menken

Department of Mathematics, Science and Arts Faculty, Mersin University, Mersin, Turkey

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ABSTRACT

In the present work the p-adic gamma function has been considered. The Volkenborn integral of the p-adic gamma function by using its Mahler expansion has been derived. Moreover, a new representation for the p-adic Euler constant has been given.

1. Introduction

The p-adic numbers introduced by the German mathematician Hensel (1897), are widely used in mathematics: in number theory, algebraic geometry, representation theory, algebraic and arithmetical dynamics, and cryptography. The p-adic numbers have been used applying fields with successfully applying in superfield theory of p-adic numbers by Vladimirov and Volovich (1984). In addition, the p-adic model of the universe, the p-adic quantum theory, the p-adic string theory such as areas occurred in physics (Volovich, 1987; Vladimirov and Volovich, 1984).

Throughout this paper, p is a fixed odd prime number and by Z_p; Q_p and C_p we denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of Q_p, respectively.

2. p-Adic gamma function

Morita (1975) defined the p-adic gamma function Γ_p by the formula

$$\Gamma_p(x) = \lim_{n\to x}(-1)^n \prod_{\substack{1\le j<n \\ (j,p)=1}} j$$

for $x \in Z_p$, where n approaches x through positive integers.

The p-adic gamma function Γ_p has a great interest and has been studied by Diamond (1977), Baesky (1981), Boyarsky (1980), and others. The relationship between some special functions and the p-adic gamma function Γ_p was investigated by Gross and Koblitz (1979), Cohen and Friedman (2008) and Shapiro (2012).

Diamond (1977) and Schikhof (1984) determined the p-adic Euler constant γ_p is defined by the following formula:

$$\gamma_p = \frac{\Gamma'_p(1)}{\Gamma_p(1)} = \Gamma'_p(1) = -\Gamma'_p(0) \quad (1)$$

It is clear that γ_p is an element of Z_p and has a limit representation in Q_p as

$$\gamma_p = \lim_{n\to\infty} p^{-n}\left(1-(-1)^p \frac{p^n!}{p^{n-1}!p^{p^{n-1}}}\right)$$

for $x \in Z_p$, the symbol $\binom{x}{n}$ is defined by $\binom{x}{0} = 1$ and

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} \qquad n \in N$$

The functions $x \to \binom{x}{n}$ ($x \in Z_p, n \in N$) form an orthonormal base of the space $C(Z_p \to Q_p\ )$with respect the norm $\|.\|_\infty$. This orthonormal base has the following property:

$$\binom{x}{n}' = \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}}{n-j}\binom{x}{j} \quad (2)$$

Mahler (1958) introduced an expansion for continuous functions of a p-adic variable using special polynomials as binomial coefficient polynomial. Means that for any $f \in C(Z_p \to Q_p)$, there exist unique elements $a_0, a_1, \dots, a_n$ of C_p such that $f(x) = \sum_{n=0}^{\infty} a_n\binom{x}{n}$, $(x \in Z_p)$.

The base $\left\{\binom{x}{n}: n \in N\right\}$ is called Mahler base of the space $C(Z_p \to Q_p)$, and the elements $\{a_n: n \in N\}$ in

* Corresponding Author.
Email Address: ozgecolakoglu@mersin.edu.tr (Ö. C. Havare)
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$f(x)=\sum_{n=0}^{\infty} a_n \binom{x}{n}$ are called Mahler coefficients of $f \in C(Z_p \to Q_p)$.

The Mahler expansion is a common method of representing the continuous functions from Z_p into a complete extension field of Q_p (Conrad, 1997). In order to compute Volkenborn integral of Γ_p efficiently for an arbitrary $x \in Z_p$ we will use the following Mahler expansions. The Mahler expansion of the p -adic gamma function Γ_p and its Mahler coefficients are determined by the following propositions:

Proposition 1: Let

$$\Gamma_p(x+1)=\sum_{n=0}^{\infty} a_n \binom{x}{n} \quad (x \in Z_p)$$

and then

$$\exp\left(x+\frac{x^p}{p}\right)\frac{1-x^p}{1-x}=\sum_{n=0}^{\infty} a_n \frac{(-1)^{n+1}}{n!} x^n \quad (x \in E)$$

holds where E is the region of convergence of the power series $\sum \frac{x^n}{n!}$.

Proposition 2: Let p be a prime number. Define rational numbers c_n by the power series expansion (Villegas, 2007)

$$\exp\left(x+\frac{x^p}{p}\right)=\sum_{n=0}^{\infty} c_n x^n$$

then for $0 \le a < p$ and $x \in Z_p$

$$\Gamma_p(-a+px)=\sum_{k=0}^{\infty} p^k c_{a+pk} (x)^k$$

where $(x)^k = x(x+1)\dots(x+k-1)$.

note that

$$(x)^k = (-1)^k \binom{-x}{k} k!.$$

3. Volkenborn integral

The Volkenborn integral was introduced in 1971 by Volkenborn in his Ph.D. dissertation and subsequently in the set of twin papers (Volkenborn, 1972; Volkenborn, 1974), a more recent treatment of the subject can be found in (Robert, 2000). The Volkenborn integral can be used for defining the p-adic log gamma functions, the p-adic Bernoulli numbers and polynomial, the p-adic zeta and L-functions. Special numbers and polynomials have played role in almost all areas of mathematics, in mathematical physics, computer science, engineering problems and other areas of science (Araci and Açikgöz, 2015; Kim et al., 2013a; 2013b; Simsek and Yardimci, 2016; Simsek, 2014; Srivastava et al., 2012).

The indefinite sum of a continuous function $f: Z_p \to C_p$ is the continuous function Sf interpolating

$$n \to \sum_{j=0}^{n-1} f(j) \quad (n \in N)$$

instead of $Sf \ \ (x \in Z_p)$ we can write

$$\lim_{n\to x} \sum_{j=0}^{n-1} f(j) = \sum_{j=0}^{x-1} f(j)$$

see Schikhof (1984) and Robert (2000).

Let f be a function from $C^1(Z_p \to Q_p)$. The Volkenborn integral of f on Z_p is defined by the formula

$$\int_{Z_p} f(x)dx = \lim_{n\to\infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = (Sf)'(0)$$

For any $f \in C^1(Z_p \to C_p)$., Volkenborn integral has following properties:

$$\int_{Z_p} f(x+1)dx - \int_{Z_p} f(x)dx = f'(0) \qquad (3)$$

$$\int_{Z_p} f(x+s)dx = (Sf)'(s) \qquad (4)$$

$$\int_{Z_p} f(-x)dx = \int_{Z_p} f(x+1)dx \qquad (5)$$

The Volkenborn integral in terms of the Mahler coefficients: Let $f=\sum_{n=0}^{\infty} a_n \binom{x}{n} \in C^1(Z_p \to C_p)$. Then

$$\int_{Z_p} f(x)dx = \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n+1} \qquad (6)$$

4. Results and discussion

In the present work we obtain the Volkenborn integral of p-adic gamma function and a new representative for the p-adic Euler constant.

In what follows, we indicate the Volkenborn integral with Mahler coefficients of p-adic gamma function:

Theorem 1: The equality holds:

$$\int_{Zp} \Gamma_p(x+1)dx = \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n+1}$$

for $x \in Z_p$, where a_n is defined by Proposition 1

Proof: Let $x \in Z_p$, $n \in N$. From Proposition 1 and (6), we get

$$\int_{Zp} \Gamma_p(x)dx = \int_{Z_p} \sum_{n=0}^{\infty} a_n \binom{x}{n} dx = \sum_{n=0}^{\infty} a_n \int_{Z_p} \binom{x}{n} dx$$

or

$$\int_{Zp} \Gamma_p(x+1)dx = \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n+1}.$$

Theorem 2: For $x \in Z_p$ and $n \in N$,

$$\int_{Zp} \Gamma_p(x)dx = \gamma_p + \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n+1}.$$

Proof: By using Eq. 3 we get

$$\int_{Z_p} \Gamma_p(x+1)dx - \int_{Z_p} \Gamma_p(x)dx = \Gamma_p{}'(0) \qquad (7)$$

If we substitute (1) and Theorem 1 in Eq. 7 then we obtain that

$$\sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n+1} - \int_{Zp} \Gamma_p(x)dx = -\gamma_p.$$

Proof of the theorem is finished.

Theorem 3: For all $x, s \in Z_p$, the following identity:

$$\int_{Zp} \Gamma_p(x+s)dx = \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_n \frac{(-1)^{n-j}}{n+1-j} \binom{s-1}{j}$$

is true.

Proof: From Eq. 4 and Proposition 1, we get

$$\int_{Zp} \Gamma_p(x+s)dx = \left(\sum_{n=0}^{\infty} a_n S\left(\binom{x-1}{j}\right)\right)'(s)$$

Note that $S\binom{x}{n} = \binom{x}{n+1}$. Therefore, we get

$$\int_{Zp} \Gamma_p(x+s)dx = \left(\sum_{n=0}^{\infty} a_n \binom{x-1}{n+1}\right)'(s)$$

By using (2) we can write as following

$$\int_{Zp} \Gamma_p(x+s)dx = \left(\sum_{n=0}^{\infty} a_n \sum_{j=0}^{n} \frac{(-1)^{n-j}}{n+1-j} \binom{x-1}{j}\right)(s)$$

or

$$\int_{Zp} \Gamma_p(x+s)dx = \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_n \frac{(-1)^{n-j}}{n+1-j} \binom{s-1}{j}.$$

In the case s= 0 in Theorem 3 we obtain the following corollary

Corollary 1: Let $x \in Z_p$. The following equality holds:

$$\int_{Zp} \Gamma_p(x)dx = \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_n \frac{(-1)^n}{n+1-j}$$

or

$$\int_{Zp} \Gamma_p(x)dx = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} a_n \frac{(-1)^n}{j+1}$$

From Theorem 2 and Corollary 1 we can write a new representation for the p-adic Euler constant:

Corollary 2: The p-adic Euler constant have the expansion (Schikhof, 1984):

$$\gamma_p = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} a_n \frac{(-1)^n}{j+1} + \sum_{n=0}^{\infty} a_n \frac{(-1)^{n+1}}{n+1}$$

Note that $f \in C^1(Z_p \to Q_p)$, $j \in \{0,1,\ldots,p-1\}$,

$$\int_{pZp} f(j+x)dx = p^{-1} \int_{Zp} f(j+px)dx$$

Theorem 4: If $x \in pZ_p$then

$$\int_{pZ_p} \Gamma_p(x)dx = -\sum_{k=0}^{\infty} c_{pk} \frac{k! p^{k-1}}{k(k+1)}$$

where c_{pk} is defined by Proposition 2.

Proof: Assume that $|x|_p < 1$. We have

$$\int_{pZ_p} \Gamma_p(x)dx = p^{-1} \int_{Z_p} \Gamma_p(px)dx$$

From Proposition 2,

$$\int_{pZ_p} \Gamma_p(x)dx = p^{-1} \int_{Z_p} \sum_{k=0}^{\infty} p^k c_{pk}(x)^k dx = \sum_{k=0}^{\infty} p^{k-1} c_{pk} \int_{Z_p} (x)^k dx$$

or

$$\int_{pZ_p} \Gamma_p(x)dx = \sum_{k=0}^{\infty} p^{k-1} c_{pk}\, k!\,(-1)^k \int_{Z_p} \binom{-x}{k} dx$$

using Eq. 5, we have

$$\int_{pZ_p} \Gamma_p(x)dx = \sum_{k=0}^{\infty} p^{k-1} c_{pk}\, k!\,(-1)^k \int_{Z_p} \binom{x+1}{k} dx$$

now compute $\int_{Z_p} \binom{x+1}{k} dx$:

$$\int_{Z_p} \binom{x+1}{k} dx = \lim_{x\to 0} \frac{\binom{x+1}{k+1}}{x} = \lim_{x\to 0} \frac{\frac{x+1x}{k+1k}\binom{x-1}{k-1}}{x} = \frac{(-1)^{k-1}}{k(k+1)}.$$

so, we obtain

$$\int_{pZ_p} \Gamma_p(x)dx = \sum_{k=0}^{\infty} p^{k-1} c_{pk}\, k!\,(-1)^k \frac{(-1)^{k-1}}{k(k+1)} = -\sum_{k=0}^{\infty} \frac{p^{k-1} c_{pk} k!}{k(k+1)}$$

Recall that $T_p = Z_p \backslash pZ_p$. From Theorem 2 and Theorem 4, we obtain following corollary.

Corollary 3: Let $x \in T_p$. Then

$$\int_{T_p} \Gamma_p(x)dx = \gamma_p + \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n+1} + \sum_{k=0}^{\infty} \frac{p^{k-1} c_{pk} k!}{k(k+1)}$$

where a_n is defined by Proposition 1 and c_{pk} is defined by Proposition 2.

5. Conclusion

In this paper, we study the p-adic Gamma function and the following results are obtained:

1. The Volkenborn integral of the p-adic Gamma function is evaluated.
2. For the p-adic Euler constant which has important role in many areas, useful representation is derived.

References

Araci S and Açikgöz M (2015). A note on the values of weighted q-Bernstein polynomials and weighted q-Genocchi numbers. Advances in Difference Equations, 2015(1): 30-39.

Baesky D (1981). On Morita's $ p $-adic $\Gamma $-function. Groupe de Travail D'analyse Ultramétrique, 5: 1977-1978.

Boyarsky M (1980). p-adic gamma functions and Dwork cohomology. Transactions of the American Mathematical Society, 257(2): 359-369.

Cohen H and Friedman E (2008). Raabe's formula for $ p $-adic gamma and zeta functions (Formules de Raabe pour les fonctions gamma et zêta $ p $-adiques). Annales de L'institut Fourier, 58(1): 363-376.

Conrad K (1997). p-adic Gamma Functions. Ph.D. Dissertations, Harvard Mathematics Department, Harvard University, Cambridge, USA.

Diamond J (1977). The p-adic log gamma function and p-adic Euler constants. Transactions of the American Mathematical Society, 233: 321-337.

Gross BH and Koblitz N (1979). Gauss sums and the p-adic Γ-function. Annals of Mathematics, 109(3): 569-581.

Hensel K (1897). Über eine neue Begründung der Theorie der algebraischen Zahlen. Jahresbericht der Deutschen Mathematiker-Vereinigung, 6: 83-88.

Kim DS, Kim T, and Seo JJ (2013a). A note on Changhee polynomials and numbers. Advanced Studies in Theoretical Physics, 7(20): 993-1003.

Kim T, Kim DS, Mansour T, Rim SH, and Schork M (2013b). Umbral calculus and Sheffer sequences of polynomials. Journal of Mathematical Physics, 54(8): 083504. https://doi.org/10.1063/1.4817853.

Mahler K (1958). An interpolation series for continuous functions of a p-adic variable. Journal Reine Angew: Math, 199: 23-34.

Morita Y (1975). A p-adic analogue of the $\Gamma $-function. Journal Ournal of the Faculty of Science, the University of Tokyo (Sect. 1 A, Mathematics), 22(2): 255-266.

Robert AM (2000). A course in p-adic analysis. volume 198 of Graduate Texts in Mathematics. Springer-Verlag, New York, USA.

Schikhof WH (1984). Ultrametric calculus: An introduction to p-adic analysis. Cambridge University Press, Cambridge, UK.

Shapiro I (2012). Frobenius map and the p-adic gamma function. Journal of Number Theory, 132(8): 1770-1779.

Simsek Y (2014). Special numbers on analytic functions. Applied Mathematics, 5(07): 1091-1098.

Simsek Y and Yardimci A (2016). Applications on the Apostol-Daehee numbers and polynomials associated with special numbers, polynomials, and p-adic integrals. Advances in Difference Equations, 2016(1): 308-322.

Srivastava HM, Kurt B, and Simsek Y (2012). Some families of Genocchi type polynomials and their interpolation functions. Integral Transforms and Special Functions, 23(12): 919-938.

Villegas FR (2007). Experimental number theory (No. 13). Oxford University Press, Oxford, UK.

Vladimirov VS and Volovich IV (1984). Superanalysis. I. Differential-Calculus. Theoretical and Mathematical Physics, 59(1): 317-335.

Volkenborn A (1972). Ein p-adisches integral und seine anwendungen I. Manuscripta Mathematica, 7(4): 341-373.

Volkenborn A (1974). Ein p-adisches integral und seine anwendungen II. Manuscripta Mathematica, 12(1): 17-46.

Volovich IV (1987). Number theory as the ultimate physical theory. No. CERN-TH-4781-87, CERN Research Institute, Geneva, Switzerland.